PSEUDO-$\mathfrak{m}$-COMPACTNESS AND $v(P \times Q)$

BY

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The present paper is a contribution to the problem when the Hewitt realcompactification commutes with products. The results published below have a sense only in the case measurable cardinals exist. But in spite of this fact the results are of some interest; they indicate certain connections with the compact case $\beta(P \times Q)$ and complete in a way papers [2], [3] by COMFORT and NEGREPONTIS.

In the first section properties are proved being equivalent to pseudo-$\mathfrak{m}$-compactness and generalizing those known for pseudocompactness ($\mathfrak{m}$ stands for the first measurable cardinal and also for the first corresponding ordinal). A space playing the same role in pseudo-$\mathfrak{m}$-compactness as real numbers in pseudocompactness is described.

The second section deals with the equality $v(P \times Q) = vP \times vQ$ in special cases. We find out in Theorem 5 that partial analogy of our realcompact case with the GLICKSBERG theorem on $\beta(P \times Q)$ (see [5], [7]) holds. Some results of this section were published without proofs in the preliminary communication [10]. Further results of this sort concerning function spaces are prepared.

All the spaces under consideration are supposed to be uniformizable Hausdorff. The terminology of [1] and [6] is used.

§ 1.

We shall need some facts about pseudo-$\mathfrak{m}$-compactness which was introduced by ISBELL in [11][1]). We shall use the following equivalent definition – see [14] (a family of sets in a space is called discrete if each point of the space has a neighborhood intersecting at most one member of the family):

Definition 1. A space P is said to be *pseudo-$\mathfrak{m}$-compact* if each discrete family of open sets in P is of nonmeasurable cardinal.

This equivalent definition (and further ones – see e.g. [8]) is, in fact,

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1) The concept of pseudo-$\aleph$-compact spaces was also introduced and investigated under a different name in a paper "Generalizations of compact and Lindelöf spaces" by Z. Frolík, published in Czech. Math. J. *9*, 172–217 (1959); that paper contains various characterizations of pseudo-$\aleph$-compact spaces including our definition. (Added in proof.)

of covering character. What we need is an analogy with defining of pseudocompactness by means of the space R of real numbers. We shall try to give such characterizations. First we shall describe a space playing similar role in pseudo-$\mathfrak{m}$-compactness as R in pseudocompactness (it is a star space from [12], p. 190).

Denote by S the following metric space: The underlying set equals to a quotient $(M \times I)/r$, where I is the closed unit interval $[0, 1]$, M is a discrete space of cardinality $\mathfrak{m}$ and the equivalence r is a union of the identity on $M \times I$ and of $M \times (0)$; the equivalence class $M \times (0)$ will be denoted by 0 or $\langle m, 0\rangle$, $m \in M$, the remaining points are pairs $\langle m, x\rangle$ with $m \in M$ and $x \in]0, 1]$. The metric d on S is defined by

$$d\langle 0, \langle m, x\rangle\rangle = d\langle\langle m, x\rangle, 0\rangle = x$$

$$d\langle\langle m, x\rangle, \langle n, y\rangle\rangle = \begin{matrix} |x-y| & \text{if } m = n \\ x+y & \text{if } m \neq n. \end{matrix}$$

Properties of the space S: S is a complete metric space with the density character $\mathfrak{m}$; it is connected and locally connected but it is not locally compact, realcompact and pseudo-$\mathfrak{m}$-compact; realcompact and pseudo-$\mathfrak{m}$-compact subspaces of S are just subspaces of nonmeasurables cardinals.

Proposition 1. *Let $\{U_m | m \in M\}$ be a discrete family of nonvoid open subsets of a space P and let $y_m \in U_m$ for each $m \in M$. Then every mapping $f: \{y_m | m \in M\} \to S$ can be continuously extended on P into S.*

Proof. (See similar assertion in [6], 3 L. 1.) Let $fy_m = \langle n_m, x_m\rangle$ where $x_m \in [0, 1]$. For each $m \in M$ there is a continuous function $f_m: P \to [0, 1]$ such that $f_m y_m = x_m$, $f_m[P - U_m] = (0)$. Define $g: P \to S$ in the following way:

$$gy = 0 \qquad \text{if } y \notin \cup \{U_m | m \in M\}$$

$$gy = \langle n_m, f_m y\rangle \text{ if } y \in U_m.$$

Since the restriction of g to $P - \cup \{U_n | n \neq m\}$ is equal to the composition of f_m and the embedding of $[0, 1]$ onto its n_m-copy in S and, hence, is continuous, the continuity of g follows from the fact that the family $\{P - \cup \{U_n | n \neq m\} | m \in M\}$ is an interior covering of P.

In the following two theorems we shall state equivalent properties of pseudo-$\mathfrak{m}$-compactness which are analogous to those of pseudocompactness for R instead of S, N instead of M and compactness instead of realcompactness. By $\beta_S P$ we mean an S-compactification of P in the sense of [4], i.e., a reflection of P in the full subcategory of closed subspaces of powers S^A and their homeomorphs. One of the copies of $\beta_S P$ can be constructed by the Čech's method (a closure of P in $S^{C(P,S)}$).

Theorem 1. *The following properties of a space P are equivalent:*

(1) *P is pseudo-$\mathfrak{m}$-compact;*

(2) *There is no copy M' of M in P such that each continuous mapping on M' into S can be continuously extended on P into S;*

(3) *If f is a continuous mapping on P into S then $f[P]$ is realcompact (i.e., $f[P]$ is of nonmeasurable cardinal);*

(4) $vP = \beta_S P$.

Proof. Let M' be a copy of M in P with the property from (2). Then there is a continuous mapping f on P into S such that $f[M'] = M \times (1)$. Thus a continuous image of P is not pseudo-$\mathfrak{m}$-compact and hence P is not pseudo-$\mathfrak{m}$-compact, too. Therefore (1) implies (2). Now we shall prove that (2) implies (3). Suppose that there is a continuous mapping f on P into S such that $f[P]$ is of measurable cardinal. We may suppose that $f[P]$ contains $M \times (1)$. By Proposition 1, $f^{-1}[M \times (1)]$ contains a copy of M with the property required in (2). The implication (3) to (4) follows at once from a construction of $\beta_S P$. Indeed, we can regard $\beta_S P$ as a closure of P in $S^{C(P,S)}$ and, hence by (3), $\beta_S P$ is realcompact; since always $\beta_S P \subset vP$, the condition (4) is fulfilled. It remains to prove that (4) implies (1). Let P be not pseudo-$\mathfrak{m}$-compact. Then there is a discrete family $\{U_m | m \in M\}$ of nonvoid open sets in P. If D is constructed so that $D \subset \cup \{U_m | m \in M\}$, $D \cap U_m$ is a one-point set for each m, then by Proposition 1, $\beta_S D$ is equal to the closure of D in $\beta_S P$ and vD is equal to the closure of D in vP. Since always $\beta_S D = D$, $vD \neq D$ the equality $\beta_S P = vP$ cannot hold in this case.

For an additional characterization of pseudo-$\mathfrak{m}$-compactness we need a concept related to that of G_δ-set. The following definition is sufficient to our purposes but, unfortunately, very complicated; we believe there exists a more convenient form of it.

Definition 2. A subset A of a space P is said to be a *$G_\delta(\mathfrak{m})$-set* in P if there exists a family $\{G_\alpha | \alpha < \mathfrak{m}\}$ of open subsets in P such that

(a) $\cap \{\bar{G}_\alpha | \alpha < \mathfrak{m}\} = A$;

(b) if $\alpha < \beta$ then $G_\alpha \supset G_\beta$ and if α is limit then $G_\alpha = \cap \{G_\beta | \beta < \alpha\}$;

(c) $\{G_\alpha - G_{\alpha+1} | \alpha < \mathfrak{m}\}$ is an open collection of measurable cardinal which is discrete in $P - A$.

Theorem 2. *A space P is pseudo-$\mathfrak{m}$-compact if and only if every $G_\delta(\mathfrak{m})$-set in vP meets P.*

Proof. If there is a $G_\delta(\mathfrak{m})$-set in vP disjoint with P then P is not pseudo-$\mathfrak{m}$-compact by (c). Thus the condition is necessary. Now, let P be not pseudo-$\mathfrak{m}$-compact. Then there is a discrete family $\{U_\alpha | \alpha < \mathfrak{m}\}$ of nonvoid open sets in P. Choose a nonvoid open V_α for each α such that V_α and $P - U_\alpha$ are functionally separated in P. Let V'_α be an open set in vP the trace of which in P is V_α and denote by G_α the open set $\cup \{V'_\beta | \alpha \leqq \beta < \mathfrak{m}\}$. We shall prove that $A = \cap \{\bar{G}_\alpha | \alpha < \mathfrak{m}\}$ is a $G_\delta(\mathfrak{m})$-set in vP disjoint with P. We need verify that the system $\{G_\alpha\}$ and A satisfy (c) and that A is disjoint with P. For the rest of the proof we identify

M with the set $T_{\mathfrak{m}}$ of ordinals smaller than $\mathfrak{m}$. First we shall prove (c). Let $x \in vP - A$; then there is a β such that $x \notin \bar{G}_\beta$. As in the proof of Proposition 1 we can construct a continuous mapping $f: P \to S$ such that $f[V_\alpha] = \langle \alpha, 1 \rangle$ for $\alpha < \beta$ and $f[P - \cup \{U_\alpha | \alpha < \beta\}] = (0)$. Extending this mapping continuously on vP we obtain a mapping g with the same image as f. Thus gx has a neighborhood in S intersecting at most one $\langle \alpha, 1 \rangle$, $\alpha < \beta$. It follows that x has a neighborhood intersecting at most one $V'_\alpha = G_\alpha - G_{\alpha+1}$, $\alpha < \beta$ and, consequently, has a neighborhood intersecting at most one $G_\alpha - G_{\alpha+1}$, $\alpha < \mathfrak{m}$. The remaining property of (c) is clear. Finally, since $P \cap \bar{G}_\alpha = \cup \{\bar{V}^P_\beta | \beta \geqq \alpha\}$, the set $P \cap A$ is empty. The proof is complete.

Remark. As it was seen in the proof we may request for $\{(G_\alpha - - G_{\alpha+1}) \cap P\}$ to be uniformly discrete in P.

§ 2.

In this section we shall be interested in the equality $v(P \times Q) = vP \times vQ$ (by this equality we mean that the continuous mapping from $v(P \times Q)$ into $vP \times vQ$, leaving all the points of $P \times Q$ fixed, is a homeomorphism onto).

The following theorem is not so important but it is of some interest. It generalizes the example 4.8 from [3] and the proposition 4.6 from [13] and entails that all members of $\mathscr{R}$ from [13] are of nonmeasurable cardinal.

Theorem 3. *Let Q be discrete. Then $v(P \times Q) = vP \times vQ$ if and only if either P or Q is of nonmeasurable cardinal.*

Proof. The case when card Q is nonmeasurable is trivial. If card Q is measurable and card P is nonmeasurable, then each continuous bounded function on $P \times Q$ can be continuously extended on $\beta P \times Q$ and, hence, on $\beta P \times vQ$ by Theorem 2.8 in [3]. Thus it remains to prove that if both card P and card Q are measurable then there is a continuous function f on $P \times Q$ which cannot be continuously extended on $vP \times vQ$. We may and shall assume that card $Q \leqq$ card P. It is easy to construct an injective mapping $h: Q \to P$ such that there is a point $q_0 \in vQ - Q$ with $\tilde{h}q_0 \notin h[Q]$, where $\tilde{h}: vQ \to vP$ is a continuous extension of h. Indeed, there is an injective mapping $k: Q \to P$ such that $k[Q] \neq P$; if k has not the property required for h we pick out $p_1 \in P - k[Q]$, $p_2 \in \tilde{k}[vQ - Q]$ and put

$$h = k_{Q-k^{-1}[p_2]} \cup \langle k^{-1}[p_2], p_1 \rangle;$$

then $q_0 \in \tilde{k}^{-1}[p_2] - Q$. Now, define $f: vP \times Q \to [0, 1]$ as follows: for every $q \in Q$, $f\langle \cdot, q \rangle$ is a continuous function on vP into $[0, 1]$ having the value 1 in hq and the value 0 in $\tilde{h}q_0$. Of course, this function f cannot be continuously extended on $vP \times vQ$ since it has the value 0 on the set $\{\langle \tilde{h}q_0, q \rangle | q \in Q\}$, the value 1 on $\{\langle hq, q \rangle | q \in Q\}$ and closures of both these sets meet in $vP \times vQ$ (they contain $\langle \tilde{h}q_0, q_0 \rangle$).

Corollary. *If card P is measurable and Q is not pseudo-$\mathfrak{m}$-compact then $v(P \times Q) \neq vP \times vQ$.*

Proof. The space Q contains a discrete subspace M of measurable cardinal such that $P \times M$ is C-embedded in $P \times Q$. Therefore, by the foregoing theorem, $v(P \times Q) \neq vP \times vQ$.

Our following assertion completes the theorem 2.2 from [2] for the case of measurable cardinals. The theorem 2.2 from [2] asserts that if Q is a locally compact realcompact space of nonmeasurable cardinal, then $v(P \times Q) = vP \times Q$ for each space P. In the proof we shall make use of the following formal modification of Shirota theorem ([6], [15]): P is realcompact if and only if it is pseudo-$\mathfrak{m}$-compact and has a complete uniformity.

Theorem 4. *Let Q be locally compact realcompact. Then $v(P \times Q) = vP \times Q$ if and only if either card Q is nonmeasurable or P is pseudo-$\mathfrak{m}$-compact.*

Proof. Since $C(P \times Q) = C(P, C(Q))$ and $C(vP \times Q) = C(vP, C(Q))$ (these equalities stand for the canonical bijections), where $C(Q)$ has the compact-open topology, the equality $v(P \times Q) = vP \times Q$ holds if and only if each continuous mapping $f: P \to C(Q)$ can be continuously extended to a mapping on vP into $C(Q)$. If Q is of nonmeasurable cardinal, then $C(Q)$ is realcompact (it has a complete uniformity). If P is pseudo-$\mathfrak{m}$-compact then $f[P]$ has the same property and so $\overline{f[P]}$ is realcompact. In both cases f can be continuously extended on vP into $C(Q)$. We have proved that our condition is sufficient. Its necessity follows immediately from Corollary of Theorem 3.

Assume for a while that P and Q are of measurable cardinals. If either Q is discrete or locally compact realcompact, then $v(P \times Q) = vP \times vQ$ if and only if $P \times Q$ is pseudo-$\mathfrak{m}$-compact. This assertion is analogous to Glicksberg theorem saying that if P and Q are infinite spaces then $\beta(P \times Q) = \beta P \times \beta Q$ if and only if $P \times Q$ is pseudocompact. As we find out in Theorem 5 and the example following it this analogy holds in one direction only.

Theorem 5. *Let P and Q be of measurable cardinals. If $v(P \times Q) = vP \times vQ$ then $P \times Q$ is pseudo-$\mathfrak{m}$-compact.*

Proof. We shall proceed similarly as Glicksberg in [7]. Let P, Q be spaces of measurable cardinals such that $v(P \times Q) = vP \times vQ$. Then, by Corollary of Theorem 3, P and Q are pseudo-$\mathfrak{m}$-compact spaces. Assume that $P \times Q$ is not pseudo-$\mathfrak{m}$-compact. Then there is a $G_\delta(\mathfrak{m})$-set A in $vP \times vQ$ disjoint with $P \times Q$ (Theorem 2) and a corresponding family $\{G_\alpha\}$ with properties stated in Definition 2 and in addition such that the collection $\{(G_\alpha - G_{\alpha+1}) \cap (P \times Q)\}$ is uniformly discrete in $P \times Q$ (the remark following Theorem 2). First we shall prove that $A \cap (P \times vQ) = \emptyset$. Let $\langle p, q \rangle \in A \cap (P \times vQ)$ and let $\{U_\alpha\}$ be a discrete system of open sets in

$P \times Q$ such that for each α the sets $(P \times Q) - U_\alpha$ and $(G_\alpha - G_{\alpha+1}) \cap (P \times Q)$ are functionally separated in $P \times Q$. Then $\{U_\alpha \cap ((p) \times Q)\}$ is a discrete system of open sets in $(p) \times Q$ of cardinality $\mathfrak{m}$. Indeed, if $\{U_\alpha \cap ((p) \times Q)\}$ had smaller cardinality than $\mathfrak{m}$, then there would be a β such that $U_\alpha \cap ((p) \times Q) = \emptyset$ for $\alpha \geqq \beta$; but in this case no continuous function on $P \times Q$ being equal to 1 on $G_\beta \cap (P \times Q)$ and 0 on $P \times Q - \cup \{U_\alpha | \alpha \geqq \beta\}$ could be continuously extended to $\langle p, q \rangle$ ($\langle p, q \rangle$ is an accumulation point of both sets $(p) \times Q$ and $G_\beta \cap (P \times Q)$). This fact (discreteness of $\{U_\alpha \cap \cap ((p) \times Q)\}$) contradicts to pseudo-$\mathfrak{m}$-compactness of Q. Now, take a subset X of $G_0 \cap (P \times Q)$ such that $X \cap (G_\alpha - G_{\alpha+1})$ is a one-point set for each α. Pick out an $x = \langle p, q \rangle \in \overline{X} - (P \times Q)$, the closure being in $vP \times vQ$. Then for each α there is a neighborhood U_α of x such that $U_\alpha \cap (P \times Q) \subset G_\alpha$ and hence $x \in A$. By the first part of the proof $p \notin P$. We shall prove that the collection $\{(G_\alpha - G_{\alpha+1}) \cap (P \times (q))\}$ is an open discrete collection in $P \times (q)$ of measurable cardinality, which contradicts to pseudo-$\mathfrak{m}$-compactness of P. It is clear that the collection is open and discrete in $P \times (q)$. Assume it has a nonmeasurable cardinal. Since $A \cap (P \times (q)) = \emptyset$ there is an $\alpha < \mathfrak{m}$ such that $G_\alpha \cap (P \times (q)) = \emptyset$. It follows there exists a continuous function defined on $P \times vQ$ with a value 1 on X and 0 on $P \times (q)$; evidently this function cannot be continuously extended to $\langle p, q \rangle$. The proof is complete.

Corollary. *If P is a locally compact realcompact space and Q is pseudo-$\mathfrak{m}$-compact, then $P \times Q$ is pseudo-$\mathfrak{m}$-compact.*

Proof follows from the preceding theorem and from Theorem 4 in the case card P and card Q are measurable. The remaining cases are trivial (if card P is nonmeasurable and Q is pseudo-$\mathfrak{m}$-compact then $P \times Q$ is pseudo-$\mathfrak{m}$-compact, too).

We do not know whether the assertion of Corollary remains true after replacing realcompact by pseudo-$\mathfrak{m}$-compact[2]).

It follows from Theorem 4 that in the preceding theorem the assumption on cardinality of P and Q cannot be omitted.

Unlike the compact case the converse of Theorem 5 does not hold. It is easy to state many examples of spaces P, Q of measurable cardinals such that $P \times Q$ is pseudo-$\mathfrak{m}$-compact and $v(P \times Q) \neq vP \times vQ$. But in all the examples we know, this nonequality is caused by factors of non-measurable cardinals. We believe that the converse of Theorem 5 is true provided these factors are excluded.

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[2]) Now we can answer in the negative the question whether realcompactness in Corollary of Theorem 5 can be replaced by pseudo-$\mathfrak{m}$-compactness. It suffices to put $P = T_\mathfrak{m}$ with the usual order-topology (i.e., P is locally compact pseudo-compact) and $Q = T'_{\mathfrak{m}+1}$, which is the set $T_{\mathfrak{m}+1}$ with a discrete topology on $T_\mathfrak{m}$ and the usual order-neighborhoods at $\mathfrak{m}$ (i.e., Q is realcompact); the product $P \times Q$ is not pseudo-$\mathfrak{m}$-compact because the cover $\{A_\alpha | -1 \leqq \alpha < \mathfrak{m}\}$, where $A_{-1} = \{\langle \beta, \gamma \rangle | \beta \leqq \gamma \leqq \mathfrak{m}\}$ and $A_\alpha = \{\langle \beta, \alpha \rangle | \beta > \alpha\}$ for $\alpha \geqq 0$, is a disjoint open (hence uniformizable) cover of $P \times Q$. (Added in proof.)

Example. Let X be a compact space of measurable cardinal and T'_{ω_1+1} be the set T_{ω_1+1} with a T_1-topology inducing a discrete topology on T_{ω_1} and having the same base of neighborhoods at ω_1 as the order topology in T_{ω_1+1}. The space T'_{ω_1+1} is realcompact and $v(T_{\omega_1} \times T'_{\omega_1+1}) \neq \neq T_{\omega_1+1} \times T'_{\omega_1+1}$. Put $P = T_{\omega_1} + X$, $Q = T'_{\omega_1+1} + X$ (the operation + means a disjoint union). Then $P \times Q$ is pseudo-$\mathfrak{m}$-compact and $v(P \times Q) \neq vP \times vQ$.

There appeared an interesting question in connection with the last example: do minimal cardinals α, β exist such that there are spaces P, Q of cardinalities α, β, respectively, with $v(P \times Q) \neq vP \times vQ$? The example above produces spaces with $\alpha = \beta = \aleph_1$. A slight modification of Example 5.3 from [13] produces spaces with $\alpha = \aleph_0$, $\beta = \exp \aleph_0$ (here $P = N \cup (x)$, $x \in \beta N - N$, and $Q = N \cup X$ is a pseudocompact subspace of βN not containing x – for existence of such a Q see e.g. [1], p. 864, 3 (f)) and, hence, shows that under the hypotheses of continuum there are spaces P, Q of cardinalities $\aleph_0$ and $\aleph_1$, respectively, such that $v(P \times Q) \neq P \times vQ$. We wonder if such spaces exist without the assumption of the hypothesis of continuum.

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